

ON THE NUMERICAL APPROXIMATION OF p -BIHARMONIC AND ∞ -BIHARMONIC FUNCTIONS

NIKOS KATZOURAKIS AND TRISTAN PRYER

ABSTRACT. In [KP16b] the authors introduced a second order variational problem in L^∞ . The associated equation, coined the ∞ -Bilaplacian, is a *third order* fully nonlinear PDE given by

$$\Delta_\infty^2 u := (\Delta u)^3 |\mathbf{D}(\Delta u)|^2 = 0.$$

In this work we build a numerical method aimed at quantifying the nature of solutions to this problem which we call ∞ -Biharmonic functions. For fixed p we design a mixed finite element scheme for the pre-limiting equation, the p -Bilaplacian

$$\Delta_p^2 u := \Delta(|\Delta u|^{p-2} \Delta u) = 0.$$

We prove convergence of the numerical solution to the weak solution of $\Delta_p^2 u = 0$ and show that we are able to pass to the limit $p \rightarrow \infty$. We perform various tests aimed at understanding the nature of solutions of $\Delta_\infty^2 u$ and in 1-d we prove convergence of our discretisation to an appropriate weak solution concept of this problem, that of \mathcal{D} -solutions.

1. INTRODUCTION AND THE ∞ -BILAPLACIAN

Let $\Omega \subset \mathbb{R}^d$ be an open and bounded set. For a given function $u : \Omega \rightarrow \mathbb{R}$ we denote the gradient of u as $\mathbf{D}u : \Omega \rightarrow \mathbb{R}^d$ and its Hessian $\mathbf{D}^2u : \Omega \rightarrow \mathbb{R}^{d \times d}$ and Laplacian $\Delta u : \Omega \rightarrow \mathbb{R}$. The p -Bilaplacian

$$(1.1) \quad \Delta_p^2 u := \Delta(|\Delta u|^{p-2} \Delta u) = 0$$

is a fourth order elliptic partial differential equation (PDE) which is a nonlinear generalisation of the Bilaplacian. Such problems typically arise from areas of elasticity, in particular, the nonlinear case can be used as a model for travelling waves in suspension bridges [LM90, GM10]. It is a fourth order analogue to its second order sibling, the p -Laplacian, and as such it is useful as a prototypical nonlinear fourth order problem.

The efficient numerical simulation of general fourth order problems has attracted growing interest. A conforming approach to this class of problems would require the use of C^1 finite elements, the Argyris element for example [Cia78, Section 6]. From a practical point of view the approach presents difficulties, in that the C^1 finite elements are difficult to design and complicated to implement, especially when working in three spatial dimensions. Other possibilities include discontinuous Galerkin methods, which form a class of nonconforming finite element method. If $p = 2$ we have the special case that the (2-)Bilaplacian, $\Delta^2 u = 0$, is linear. It has been well studied in the context of both C^1 finite elements [Cia78] and discontinuous Galerkin methods; for example, the papers [LS03, GH09] study the use of h - k dG finite elements (where k here means the local polynomial degree as opposed to the usual convention which is p) applied to the (2-)Bilaplacian. The numerical approximation of p -Bilaplacian (quasi-linear, fourth order) type PDEs is relatively untouched. To the authors' knowledge, the only known work is [Pry14] where a discontinuous Galerkin method based on a variational principle was derived and was shown to converge under minimal regularity. However, no rates of convergence were proven.

In this work we propose a method based on C^0 -mixed finite elements. We rewrite the minimisation problem in the spirit of a saddle point problem and prove that the method converges under minimal regularity of the solution. In addition, using an inf-sup condition and tools from [San93, Far98, GR12] we are able to show

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that under additional regularity assumptions the approximation converges with specific rates that depend on p .

Making use of these convergence results and the uniqueness of solutions in one dimension we are able to justify that approximations of the p -Bilaplacian for large p are “good” approximations to ∞ -Biharmonic functions. These functions are solutions of the ∞ -Bilaplacian which is the PDE

$$(1.2) \quad \Delta_\infty^2 u := (\Delta u)^3 |D(\Delta u)|^2 = 0.$$

It was derived in [KP16b] as the formal limit of the p -Bilaplacian (1.1) as $p \rightarrow \infty$. The ∞ -Bilaplacian is the prototypical example of a PDE from 2nd order Calculus of Variations in L^∞ , arising as the analogue of the *Euler–Lagrange* equation associated with critical points of the supremal functional

$$(1.3) \quad \mathcal{J}[u; \infty] := \|\Delta u\|_{L^\infty(\Omega)}.$$

Variational problems in L^∞ are notoriously challenging. The 1st order case is reasonably well understood and was initiated in the sequence of works by Aronsson [Aro65, Aro66, Aro68, Aro69, c.f.]. In this case, the respective Euler-Lagrange equation associated with critical points of the functional

$$(1.4) \quad \mathcal{J}[u] = \|Du\|_{L^\infty(\Omega)}$$

is quasi-linear, 2nd order and given by

$$(1.5) \quad \Delta_\infty u = (Du \otimes Du) : D^2 u = 0.$$

This equation is called the ∞ -Laplacian and can be derived through a p -approximation of the underlying $W^{1,p}$ energy functional, see [Pry16, KP16a].

It can be easily seen that solutions to (1.2) can not in general be C^3 not even when $d = 1$; in particular, the Dirichlet problem is not solvable in the class of classical solutions. For a more extensive discussion we refer to [KP16b]. Hence, the development of a solution concept which can be interpreted in an appropriate weak sense is in order. In the case of the ∞ -Laplacian, the appropriate notion is that of the Crandall-Ishii-Lions notion of viscosity solutions. For an introduction to this theory we refer to the monograph [Kat15]. We note that in the framework of viscosity solutions we can obtain uniqueness of solution for the Dirichlet problem [Jen93]. In the case of 2nd order Calculus of Variations in L^∞ the viscosity solution concept for the resulting equations is no longer applicable since we do not have access to a maximum principle for 3rd order PDEs like (1.2), from which the solution concept stems.

One possibility for a generalised solution concept to (1.2) is that of \mathcal{D} -solutions [Kat16b, Kat16a, KP16b]. Roughly, this is a probabilistic approach where derivatives that do not exist classically are represented as limits of difference quotients into Young measures over a compactification of the space of derivatives. This solution concept has already borne substantial fruit in the 1st order vectorial case of Calculus of Variations in L^∞ , as well as for more general PDE systems. In the present 2nd order setting it proves to be an appropriate notion as well, since absolute minimisers $u \in \mathcal{W}_g^{2,\infty}(\Omega)$ satisfying

$$(1.6) \quad \|\Delta u\|_{L^\infty(\Omega')} \leq \|\Delta v\|_{L^\infty(\Omega')} \quad \forall \Omega' \Subset \Omega \text{ and } v \in \mathcal{W}_u^{2,\infty}(\Omega'),$$

are indeed unique \mathcal{D} -solutions of (1.2), at least for $d = 1$. Note that the appropriate space to take minimisers is *not* $\mathcal{W}_g^{2,\infty}(\Omega)$ but rather the larger space

$$(1.7) \quad \mathcal{W}_g^{2,\infty}(\Omega) := \left\{ u \in \bigcap_{p \in (1,\infty)} W_g^{2,p}(\Omega) : \Delta u \in L^\infty(\Omega) \right\}.$$

Uniqueness for the Dirichlet problem amongst this solution class is, for $d > 1$, still an open problem at the time of writing this¹. In [KP16b] it has been shown that in one spatial dimension the problem does indeed have a unique absolutely minimising \mathcal{D} -solution, while we also have uniqueness in the subclass of C^3 functions for all spatial dimensions.

The design of numerical schemes that are compatible with these duality-free solution concepts is extremely difficult. Even for the well developed area of viscosity solutions most numerical schemes that exist which are compatible with the solution concept are based on the arguments of [BS91] which advocates approximations

¹In an upcoming paper of the first author with R. Moser it is established that uniqueness of ∞ -Biharmonic functions with prescribed boundary values indeed holds in all spatial dimensions $d > 1$.

based on differences and which satisfy a discrete monotonicity property. The only other methodology in the design of numerical schemes for the ∞ -Laplacian is to make use of the variational principle from which the equation is derived. Galerkin approximations of the p -Laplacian can then be shown to converge to the viscosity solution of the ∞ -Laplacian [Pry16]. This method has also been used to characterise the nature of solutions to the variational ∞ -Laplace system [KP16a]. This is also the approach we use here. We build a scheme convergent to the weak solution of the p -Bilaplacian and then justify its use as an approximation of ∞ -Biharmonic functions.

The rest of the paper is set out as follows: In §2 we formalise notation and begin exploring some of the properties of the p -Bilaplacian. In particular, we reformulate the PDE as a saddle point type problem. We show inf-sup conditions for the underlying operators guarantee that the saddle point type problem is well posed, motivating the discretisation of this directly. In §3 we perform the discretisation for fixed p and show that discrete versions of the inf-sup conditions hold. A priori results for both primal and auxiliary variables are a consequence of this. Numerical experiments are given in §4 illustrating the behaviour of numerical approximations to this problem. In addition, we examine the solutions for large p and make various conjectures as to the structure of solutions in multiple spatial dimensions.

2. APPROXIMATION VIA THE p -BILAPLACIAN

In this section we describe how ∞ -Biharmonic functions can be approximated using p -Biharmonic functions. We give a brief introduction to the p -Bilaplacian problem, beginning by introducing the Sobolev spaces

(2.1)

$$L^p(\Omega) = \left\{ \phi \text{ measurable} : \int_{\Omega} |\phi|^p \, d\mathbf{x} < \infty \right\} \text{ for } p \in [1, \infty) \text{ and } L^\infty(\Omega) = \{ \phi \text{ measurable} : \text{ess sup}_{\Omega} |\phi| < \infty \},$$

$$(2.2) \quad W^{l,p}(\Omega) = \{ \phi \in L^p(\Omega) : D^\alpha \phi \in L^p(\Omega), \text{ for } |\alpha| \leq l \} \text{ and } H^l(\Omega) := W^{l,2}(\Omega),$$

which are equipped with the following norms and semi-norms:

$$(2.3) \quad \|v\|_{L^p(\Omega)}^p := \int_{\Omega} |v|^p \, d\mathbf{x} \text{ for } p \in [1, \infty) \text{ and } \|v\|_{L^\infty(\Omega)} := \text{ess sup}_{\Omega} |v|$$

$$(2.4) \quad \|v\|_{l,p}^p := \|v\|_{W^{l,p}(\Omega)}^p = \sum_{|\alpha| \leq l} \|D^\alpha v\|_{L^p(\Omega)}^p$$

$$(2.5) \quad |v|_{l,p}^p := |v|_{W^{l,p}(\Omega)}^p = \sum_{|\alpha|=l} \|D^\alpha v\|_{L^p(\Omega)}^p$$

where $\alpha = \{\alpha_1, \dots, \alpha_d\}$ is a multi-index, $|\alpha| = \sum_{i=1}^d \alpha_i$ and derivatives D^α are understood in the weak sense. We pay particular attention to the case $l = 2$ and define

$$(2.6) \quad W_g^{2,p}(\Omega) := g + W_0^{2,p}(\Omega) = \{ \phi \in W^{2,p}(\Omega) : \phi|_{\partial\Omega} = g \text{ and } D\phi|_{\partial\Omega} = Dg \},$$

for a prescribed function $g \in W^{2,\infty}(\Omega)$, where the boundary condition is understood in the trace sense if $\partial\Omega \in C^{0,1}(\Omega)$. We note that if $p > d$, then the boundary condition is satisfied in the pointwise sense since $W_0^{2,p}(\Omega) \subseteq C^1(\bar{\Omega})$.

For the p -Bilaplacian, the action functional is given as

$$(2.7) \quad \mathcal{J}[u; p] = \int_{\Omega} |\Delta u|^p \, d\mathbf{x}. \quad ^2$$

We then look to find a minimiser over the space $W_g^{2,p}(\Omega)$, that is, to find $u \in W_g^{2,p}(\Omega)$ such that

$$(2.8) \quad \mathcal{J}[u; p] = \min_{v \in W_g^{2,p}(\Omega)} \mathcal{J}[v; p].$$

²Typically $\mathcal{J}[u; p] = \frac{1}{p} \int |\Delta u|^p$. Note here the rescaling has no effect on the resultant Euler–Lagrange equations as the Lagrangian is independent of u .

If we assume temporarily that we have access to a smooth minimiser, i.e., $u \in C^4(\Omega)$, then, given that the Lagrangian is of second order, we have that the Euler–Lagrange equations are (in general) fourth order and read

$$(2.9) \quad \Delta(|\Delta u|^{p-2} \Delta u) = 0.$$

Note that, for $p = 2$, the PDE reduces to the Bilaplacian $\Delta^2 u = 0$. In general, the Dirichlet problem for the p -Bilaplacian is, given $g \in W^{2,\infty}(\Omega)$, to find u such that

$$(2.10) \quad \begin{cases} \Delta_p u := \Delta(|\Delta u|^{p-2} \Delta u) = 0, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \\ Du = Dg, & \text{on } \partial\Omega. \end{cases}$$

2.1. Definition (weak solution). The problem (2.10) has a weak formulation. Consider the semilinear form

$$(2.11) \quad \mathcal{A}(u, v) := \int_{\Omega} (|\Delta u|^{p-2} \Delta u) \Delta v \, d\mathbf{x}.$$

Then, $u \in W_g^{2,p}(\Omega)$ is a *weak solution* of (2.10) if it satisfies

$$(2.12) \quad \mathcal{A}(u, v) = 0 \quad \forall v \in W_0^{2,p}(\Omega).$$

2.2. Proposition (coercivity of \mathcal{J}). Suppose that $u \in W_0^{2,p}(\Omega)$ and $f \in L^q(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$. We have that the action functional $\mathcal{J}[\cdot; p]$ is coercive over $W_0^{2,p}(\Omega)$, that is,

$$(2.13) \quad \mathcal{J}[u; p] \geq C |u|_{2,p}^p - \gamma,$$

for some $C > 0$ and $\gamma \geq 0$. Equivalently, we have that there exists a constant $C > 0$ such that

$$(2.14) \quad \mathcal{A}(v, v) \geq C |v|_{2,p}^p \quad \forall v \in W_0^{2,p}(\Omega).$$

2.3. Corollary (weak lower semicontinuity). The action functional \mathcal{J} is weakly lower semi-continuous over $W_g^{2,p}(\Omega)$. That is, given a sequence of functions $\{u_j\}_{j \in \mathbb{N}}$ which has a weak limit $u \in W_g^{2,p}(\Omega)$, we have

$$(2.15) \quad \mathcal{J}[u; p] \leq \liminf_{j \rightarrow \infty} \mathcal{J}[u_j; p].$$

Proof The proof of this fact is a straightforward extension of [Eva98, Section 8.2 Thm 1] to second order Lagrangians, noting that \mathcal{J} is coercive (from Proposition 2.2) and convex. We omit the full details for brevity. \square

2.4. Corollary (existence and uniqueness). There exists a unique minimiser to the p -Dirichlet energy functional. Equivalently, there exists a unique (weak) solution $u \in W_g^{2,p}(\Omega)$ to the (weak form of the) Euler–Lagrange equations:

$$(2.16) \quad \int_{\Omega} |\Delta u|^{p-2} \Delta u \Delta \phi \, d\mathbf{x} = 0 \quad \forall \phi \in W_0^{2,p}(\Omega).$$

Proof Again, the result can be deduced by extending the arguments in [Eva98, Section 8.2] or [Cia78, Thm 5.3.1], again, noting the results of Propositions 2.2 and convexity. The full argument is omitted for brevity. \square

2.5. Remark (elementary properties). We will throughout this exposition use the notation p to denote the exponent appearing in the Lagrangian and q its conjugate exponent which satisfies

$$(2.17) \quad \frac{1}{p} + \frac{1}{q} = 1.$$

We shall now state some useful facts involving these numbers:

(1) For a given $v \in L^p(\Omega)$ it holds that

$$(2.18) \quad \left\| |v|^{p-1} \right\|_{L^q(\Omega)} = \|v\|_{L^p(\Omega)}^{p-1}.$$

2.6. Proposition (Poincaré inequality). *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. For any $p \in [1, \infty]$, there exists a constant $C = C(\Omega, p) > 0$ depending only on Ω and p such that*

$$(2.19) \quad \|u\|_{L^p(\Omega)} \leq C(\Omega, p) \|Du\|_{L^p(\Omega)},$$

for all $u \in W_0^{1,p}(\Omega)$.

2.7. Proposition (Calderon-Zygmund estimate [GT83, Cor 9.10]). *Let $\Omega \subset \mathbb{R}^d$ be a domain. Then, for any $p \in (1, \infty)$, there is a constant $C = C(d, p) > 0$ depending only on d and p such that*

$$(2.20) \quad \|D^2u\|_{L^p(\Omega)} \leq C(d, p) \|\Delta u\|_{L^p(\Omega)},$$

for all $u \in W_0^{2,p}(\Omega)$.

An immediate consequence of Propositions 2.6 and 2.7 above is that the norm $\|\cdot\|_{2,p}$ is equivalent to either of the seminorms $\|D^2(\cdot)\|_{L^p(\Omega)}$ and $\|\Delta(\cdot)\|_{L^p(\Omega)}$ over the space $W_0^{2,p}(\Omega)$.

2.8. Saddle point formulation of the p -Bilaplacian. The mixed formulation we propose to analyse is based on the observation that if $\phi(t) = |t|^{p-2}t$, the inverse is well defined as $\phi^{-1}(t) = \text{sgn}(t)|t|^{1/(p-1)} = |t|^{q-2}t$. Using this we make the following choice of auxiliary variable

$$(2.21) \quad w = |\Delta u|^{p-2} \Delta u$$

from which we can infer that

$$(2.22) \quad |w|^{q-2} w = \Delta u.$$

This allows us to write the problem as the mixed system:

$$(2.23) \quad \begin{cases} -\Delta u = |w|^{q-2} w, \\ -\Delta w = 0. \end{cases}$$

The mixed formulation can be written weakly as: Find a pair $(u, w) \in W_g^{1,p}(\Omega) \times W^{1,q}(\Omega)$ such that

$$(2.24) \quad \begin{cases} a(w, \psi) + b(u, \psi) = f(\psi), \\ b(w, \phi) = 0, \end{cases} \quad \forall (\psi, \phi) \in W^{1,q}(\Omega) \times W_0^{1,p}(\Omega),$$

where the semilinear form $a(w, v)$ and bilinear form $b(u, v)$ are given by

$$(2.25) \quad \begin{cases} a(w, \psi) := \int_{\Omega} |w|^{q-2} w \psi \, d\mathbf{x} \\ b(u, \psi) := \int_{\Omega} Du \cdot D\psi \, d\mathbf{x} \end{cases}$$

and the problem data

$$(2.26) \quad f(\psi) := \int_{\partial\Omega} Dg \cdot \mathbf{n} \psi \, ds$$

represents the contribution from the Neumann boundary conditions. Notice that the problem (1.1) has been reformulated in a saddle point form. Although we already know that the problem has a unique solution as a consequence of Corollary 2.4, we will show that the equivalent saddle point problem also admits a unique solution since the methodology will be useful in the sequel. We begin by recalling the following result that we will utilise.

2.9. Theorem (Generalised Gårding inequality [Sim72, Thm 6.3]). *If the boundary $\partial\Omega$ is C^1 , there exists a constant $C > 0$, such that, for all $u_0 \in W_0^{1,p}(\Omega)$ we have*

$$(2.27) \quad \|Du_0\|_{L^p(\Omega)} \leq C \left[\sup_{0 \neq v \in W_0^{1,q}(\Omega)} \frac{b(u_0, v)}{\|Dv\|_{L^q(\Omega)}} + \|u_0\|_{L^p(\Omega)} \right].$$

2.10. **Corollary** (Inf-sup stability of $b(\cdot, \cdot)$). *For any $u_0 \in W_0^{1,p}(\Omega)$, the bilinear form $b(\cdot, \cdot)$ satisfies the following inf-sup property:*

$$(2.28) \quad \|Du_0\|_{L^p(\Omega)} \leq C \sup_{0 \neq v \in W_0^{1,q}(\Omega)} \frac{b(u_0, v)}{\|Dv\|_{L^q(\Omega)}}.$$

Proof Consider the Dirichlet problem

$$(2.29) \quad \begin{cases} -\Delta z = |u_0|^{p-2}u_0, & \text{in } \Omega, \\ z = 0, & \text{on } \partial\Omega. \end{cases}$$

Then, since $u_0 \in W_0^{1,p}(\Omega)$, we have that $|u_0|^{p-2}u_0 \in L^q(\Omega)$ and the gradient estimate

$$(2.30) \quad \|Dz\|_{L^q(\Omega)} \leq C \| |u_0|^{p-1} \|_{L^q(\Omega)}.$$

Then

$$(2.31) \quad \begin{aligned} \|u_0\|_{L^p(\Omega)}^p &= \int_{\Omega} (|u_0|^{p-2}u_0)u_0 \, d\mathbf{x} \\ &= - \int_{\Omega} \Delta z \, u_0 \, d\mathbf{x} \\ &= b(u_0, z), \end{aligned}$$

through the definition of the auxiliary problem and an integration by parts. Now, making use of the estimate (2.30) we have

$$(2.32) \quad \begin{aligned} \|u_0\|_{L^p(\Omega)}^p &= \frac{b(u_0, z) \|Dz\|_{L^q(\Omega)}}{\|Dz\|_{L^q(\Omega)}} \\ &\leq \frac{b(u_0, z)}{\|Dz\|_{L^q(\Omega)}} C \| |u_0|^{p-1} \|_{L^q(\Omega)} \\ &\leq \frac{b(u_0, z)}{\|Dz\|_{L^q(\Omega)}} C \|u_0\|_{L^p(\Omega)}^{p-1} \\ &\leq C \sup_{0 \neq v \in W_0^{1,q}(\Omega)} \frac{b(u_0, v)}{\|Dv\|_{L^q(\Omega)}} \|u_0\|_{L^p(\Omega)}^{p-1}. \end{aligned}$$

Hence

$$(2.33) \quad \|u_0\|_{L^p(\Omega)} \leq C \sup_{0 \neq v \in W_0^{1,q}(\Omega)} \frac{b(u_0, v)}{\|Dv\|_{L^q(\Omega)}},$$

which combining with the result in Theorem 2.9 yields the desired conclusion. \square

2.11. **Theorem** (The mixed formulation is well posed). *For every $g \in W^{2,\infty}(\Omega)$, there exists a unique pair (u, w) solving (2.24) that satisfies*

$$(2.34) \quad \|Du\|_{L^p(\Omega)} + \|w\|_{L^q(\Omega)}^{q-1} \leq C (\|\Delta g\|_{L^p(\Omega)} + \|Dg\|_{L^p(\Omega)}).$$

Proof The results of Lemma 2.10 show that for $u_0 := u - g \in W_0^{1,p}(\Omega)$ we have

$$(2.35) \quad \begin{aligned} \|Du_0\|_{L^p(\Omega)} &\leq \sup_{0 \neq v \in W_0^{1,q}(\Omega)} \frac{b(u_0, v)}{\|Dv\|_{L^q(\Omega)}} \\ &\leq \sup_{0 \neq v \in W_0^{1,q}(\Omega)} \frac{b(u, v)}{\|Dv\|_{L^q(\Omega)}} + \sup_{0 \neq v \in W_0^{1,q}(\Omega)} \frac{b(g, v)}{\|Dv\|_{L^q(\Omega)}} \\ &\leq \sup_{0 \neq v \in W_0^{1,q}(\Omega)} \frac{-a(w, v)}{\|Dv\|_{L^q(\Omega)}} + \sup_{0 \neq v \in W_0^{1,q}(\Omega)} \frac{b(g, v)}{\|Dv\|_{L^q(\Omega)}}, \end{aligned}$$

in view of the compactness of the support of v . Now, by using Remark 2.5 and Proposition 2.6 we estimate

$$\begin{aligned}
(2.36) \quad \|Du_0\|_{L^p(\Omega)} &\leq \sup_{0 \neq v \in W_0^{1,q}(\Omega)} \frac{\|w^{q-1}\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}}{\|Dv\|_{L^q(\Omega)}} + \|Dg\|_{L^p(\Omega)} \\
&\leq C \sup_{0 \neq v \in W_0^{1,q}(\Omega)} \frac{\|w\|_{L^q(\Omega)}^{q-1} \|Dv\|_{L^q(\Omega)}}{\|Dv\|_{L^q(\Omega)}} + \|Dg\|_{L^p(\Omega)} \\
&\leq C \left(\|w\|_{L^q(\Omega)}^{q-1} + \|Dg\|_{L^p(\Omega)} \right).
\end{aligned}$$

Now take $\psi = w$ in (2.24) then

$$(2.37) \quad a(w, w) + b(u, w) = f(w).$$

Set $\phi = u_0$ in (2.24) then

$$(2.38) \quad b(w, u_0) = 0$$

and in particular

$$(2.39) \quad a(w, w) + b(u, w) - b(w, u_0) = f(w).$$

Noticing that $b(\cdot, \cdot)$ is symmetric then

$$(2.40) \quad a(w, w) + b(g, w) = f(w),$$

or explicitly

$$(2.41) \quad \int_{\Omega} |w|^q + Dg \cdot Dw \, d\mathbf{x} = \int_{\partial\Omega} Dg \cdot \mathbf{n} \, ds.$$

Integrating by parts then shows that

$$\begin{aligned}
(2.42) \quad \|w\|_{L^q(\Omega)}^q &= \int_{\Omega} \Delta g w \, d\mathbf{x} \\
&\leq \|\Delta g\|_{L^p(\Omega)} \|w\|_{L^q(\Omega)}.
\end{aligned}$$

Hence

$$(2.43) \quad \|w\|_{L^q(\Omega)}^{q-1} \leq \|\Delta g\|_{L^p(\Omega)},$$

which yields the desired result upon noting

$$(2.44) \quad \|Du\|_{L^p(\Omega)} \leq \|Du_0\|_{L^p(\Omega)} + \|Dg\|_{L^p(\Omega)}$$

and combining with (2.36). □

2.12. Theorem (the limit as $p \rightarrow \infty$). *Consider the Sobolev space (1.7) and let $(u_p)_1^\infty$ denote a sequence of weak solutions $u_p \in W_g^{2,p}(\Omega)$ to the p -Bilaplacian. Then, there exists a subsequence converging uniformly together with their derivatives to a (candidate ∞ -Biharmonic) function $u_\infty \in \mathcal{W}_g^{2,\infty}(\Omega)$. Namely,*

$$(2.45) \quad u_{p_j} \rightarrow u_\infty \text{ in } C^1(\overline{\Omega}),$$

along a subsequence as $p \rightarrow \infty$.

Proof Let $u_p \in W_g^{2,p}(\Omega)$ denote the weak solution of (2.10). In view of Corollary 2.4, we know that u_p minimises the energy functional

$$(2.46) \quad \mathcal{J}[u_p] = \int_{\Omega} |\Delta u_p|^p.$$

In particular,

$$(2.47) \quad \mathcal{J}[u_p] \leq \mathcal{J}[g],$$

where $g \in W^{2,\infty}(\Omega)$ is the associated boundary data to (2.10). Using this fact, we have

$$(2.48) \quad \|\Delta u_p\|_{L^p(\Omega)}^p = \mathcal{J}[u_p] \leq \mathcal{J}[g] = \|\Delta g\|_{L^p(\Omega)}^p,$$

and we may infer that

$$(2.49) \quad \|\Delta u_p\|_{L^p(\Omega)} \leq \|\Delta g\|_{L^p(\Omega)}.$$

Now fix a $k > d$ and take $p \geq k$. Then, by using Hölder's inequality with $r = \frac{p}{k}$ and $q = \frac{r-1}{r}$ such that $\frac{1}{r} + \frac{1}{q} = 1$, we obtain

$$(2.50) \quad \|\Delta u_p\|_{L^k(\Omega)}^k = \int_{\Omega} |\Delta u_p|^k \leq \left(\int_{\Omega} 1^q \right)^{1/q} \left(\int_{\Omega} |\Delta u_p|^p \right)^{1/r}.$$

Hence

$$(2.51) \quad \|\Delta u_p\|_{L^k(\Omega)}^k \leq |\Omega|^{\frac{r}{r-1}} \|\Delta u_p\|_{L^p(\Omega)}^k = |\Omega|^{1-\frac{k}{p}} \|\Delta u_p\|_{L^p(\Omega)}^k$$

and we see

$$(2.52) \quad \|\Delta u_p\|_{L^k(\Omega)} \leq |\Omega|^{\frac{1}{k}-\frac{1}{p}} \|\Delta u_p\|_{L^p(\Omega)}.$$

By using the triangle inequality, a double application of the Poincaré inequality (since both $u = g$ and $Du = Dg$ on $\partial\Omega$) from Proposition 2.6 and the Calderon-Zygmund L^k estimates from Proposition 2.7, we have

$$(2.53) \quad \begin{aligned} \|u_p\|_{L^k(\Omega)} &\leq \|u_p - g\|_{L^k(\Omega)} + \|g\|_{L^k(\Omega)} \\ &\leq C'(k, \Omega) \|D^2 u_p - D^2 g\|_{L^k(\Omega)} + \|g\|_{L^k(\Omega)} \\ &\leq C(k, \Omega) \|\Delta u_p - \Delta g\|_{L^k(\Omega)} + \|g\|_{L^k(\Omega)}. \end{aligned}$$

By utilising the triangle inequality again, we have

$$(2.54) \quad \begin{aligned} \|u_p\|_{L^k(\Omega)} &\leq C \left(\|\Delta u_p\|_{L^k(\Omega)} + \|g\|_{W^{2,k}(\Omega)} \right) \\ &\leq C \left(|\Omega|^{\frac{1}{k}-\frac{1}{p}} \|\Delta u_p\|_{L^p(\Omega)} + \|g\|_{W^{2,k}(\Omega)} \right), \end{aligned}$$

by virtue of (2.52). Similarly, one may show that

$$(2.55) \quad \|Du_p\|_{L^k(\Omega)} \leq C \left(|\Omega|^{\frac{1}{k}-\frac{1}{p}} \|\Delta u_p\|_{L^p(\Omega)} + \|g\|_{W^{2,k}(\Omega)} \right).$$

Thus, in view of (2.49) we infer that

$$(2.56) \quad \|u_p\|_{W^{2,k}(\Omega)} \leq C \|g\|_{W^{2,k}(\Omega)}.$$

This means that for any $k > d$ we have the uniform bound

$$(2.57) \quad \sup_{p \geq k} \|u_p\|_{W^{2,k}(\Omega)} \leq C = C(k, \Omega).$$

By invoking standard weak compactness arguments, we may extract a sub-sequence $\{u_{p_j}\}_{j=1}^{\infty} \subset \{u_p\}_{p=1}^{\infty}$ and a function $u_{\infty} \in W^{2,k}(\Omega)$ such that, for any $k > n$,

$$(2.58) \quad u_{p_j} \rightharpoonup u_{\infty} \text{ weakly in } W^{2,k}(\Omega)$$

as $j \rightarrow \infty$ and

$$(2.59) \quad \begin{aligned} \|u_{\infty}\|_{W^{2,k}(\Omega)} &\leq \liminf_{j \rightarrow \infty} \|u_{p_j}\|_{W^{2,k}(\Omega)} \\ &\leq \liminf_{j \rightarrow \infty} C \|g\|_{W^{2,k}(\Omega)}. \end{aligned}$$

Since this is true for any fixed k , it is clear that $u_{\infty} \in \bigcap_{k \in (1, \infty)} W^{2,k}(\Omega)$. Further, by the weak lower semi-continuity of the L^k norm, from (2.52) we may infer $\Delta u_{\infty} \in L^{\infty}(\Omega)$ and hence $u_{\infty} \in \mathcal{W}_g^{2,\infty}(\Omega)$, therefore concluding the proof. \square

2.13. Remark (uniqueness of “weak” solutions to the ∞ -Bilaplacian). Theorem 2.12 only guarantees convergence to a candidate ∞ -Harmonic function. If $d > 1$ it is not known if the limiting problem

$$(2.60) \quad \begin{cases} (\Delta u)^3 |\mathbf{D}(\Delta u)|^2 = 0, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \\ \mathbf{D}u = \mathbf{D}g, & \text{on } \partial\Omega, \end{cases}$$

has a unique solution. Hence, Theorem 2.12 only guarantees convergence to some candidate ∞ -Biharmonic function. If $d = 1$ the problem reduces to

$$\begin{cases} (u'')^3 (u''')^2 = 0, & \text{in } (a, b) \\ u(a) = g(a), & u(b) = g(b), \\ u'(a) = g'(a), & u'(b) = g'(b). \end{cases}$$

It has been shown that for this case there is a unique \mathcal{D} -solution [KP16b]. The solution is interpreted in a weak sense and is the only candidate ∞ -Harmonic function. This means for $d = 1$ that Theorem 2.12 guarantees convergence of the sequence of p -Biharmonic functions to the unique ∞ -Biharmonic function. (However, in an upcoming paper of the first author with R. Moser it will be shown that uniqueness of \mathcal{D} -solutions for (2.60) indeed holds in all dimensions. Notwithstanding, herein we do not attempt to study this problem further and we leave the case $d > 1$ of (2.60) for future work.)

3. DISCRETISATION OF THE p -BILAPLACIAN

In this section we describe a mixed finite element discretisation of the p -Bilaplacian. Let \mathcal{T} be a conforming triangulation of Ω , namely, \mathcal{T} is a finite family of sets such that

- (1) $K \in \mathcal{T}$ implies K is an open simplex (segment for $d = 1$, triangle for $d = 2$, tetrahedron for $d = 3$),
- (2) for any $K, J \in \mathcal{T}$ we have that $\bar{K} \cap \bar{J}$ is a full lower-dimensional simplex (i.e., it is either \emptyset , a vertex, an edge, a face, or the whole of \bar{K} and \bar{J}) of both \bar{K} and \bar{J} and
- (3) $\bigcup_{K \in \mathcal{T}} \bar{K} = \bar{\Omega}$.

The shape regularity constant of \mathcal{T} is defined as the number

$$(3.1) \quad \mu(\mathcal{T}) := \inf_{K \in \mathcal{T}} \frac{\rho_K}{h_K},$$

where ρ_K is the radius of the largest ball contained inside K and h_K is the diameter of K . An indexed family of triangulations $\{\mathcal{T}^n\}_n$ is called *shape regular* if

$$(3.2) \quad \mu := \inf_n \mu(\mathcal{T}^n) > 0.$$

Further, we define $h : \Omega \rightarrow \mathbb{R}$ to be the piecewise constant *meshsize function* of \mathcal{T} given by

$$(3.3) \quad h(\mathbf{x}) := \max_{\bar{K} \ni \mathbf{x}} h_K.$$

A mesh is called *quasi-uniform* when there exists a positive constant C such that $\max_{x \in \Omega} h \leq C \min_{x \in \Omega} h$. In what follows we shall assume that all triangulations are shape-regular and quasi-uniform although the results may be extendable even in the non-quasi-uniform case using techniques developed in [DK08].

We let \mathcal{E} be the skeleton (set of common interfaces) of the triangulation \mathcal{T} and say $e \in \mathcal{E}$ if e is on the interior of Ω and $e \in \partial\Omega$ if e lies on the boundary $\partial\Omega$ and set h_e to be the diameter of e .

We let $\mathbb{P}^k(\mathcal{T})$ denote the space of piecewise polynomials of degree k over the triangulation \mathcal{T} , i.e.,

$$(3.4) \quad \mathbb{P}^k(\mathcal{T}) = \{\phi \text{ such that } \phi|_K \in \mathbb{P}^k(K)\}$$

and introduce the *finite element space*

$$(3.5) \quad \mathbb{V} := \mathbb{P}^k(\mathcal{T}) \cap C^0(\Omega)$$

to be the usual space of continuous piecewise polynomial functions.

3.1. Definition (Ritz projection operator). The Ritz projection operator $R : W_0^{1,2}(\Omega) \rightarrow \mathbb{V}$ is defined for $v \in W_0^{1,2}(\Omega)$ as

$$(3.6) \quad \int_{\Omega} D(Rv) \cdot D\Phi = \int_{\Omega} Dv \cdot D\Phi \quad \forall \Phi \in \mathbb{V}.$$

It is well known that this operator satisfies the following approximation properties: for any $v \in W^{k+1,q}(\Omega)$,

$$(3.7) \quad \|v - Rv\|_{L^q(\Omega)} = Ch^{k+1} |v|_{k+1,q},$$

$$(3.8) \quad \|Dv - D(Rv)\|_{L^q(\Omega)} = Ch^k |v|_{k+1,q}.$$

3.2. Galerkin discretisation. Consider the space

$$(3.9) \quad \mathbb{V}_g := \{\phi \in \mathbb{V} : \phi|_{\partial\Omega} = Rg\}.$$

Then, we consider the Galerkin discretisation of (2.10), to find $(u_h, w_h) \in \mathbb{V}_g \times \mathbb{V}$ such that

$$(3.10) \quad \begin{aligned} a(w_h, \psi) + b(u_h, \psi) &= f(\psi), \\ b(w_h, \phi) &= 0, \quad \forall (\psi, \phi) \in \mathbb{V} \times \mathbb{V}_0, \end{aligned}$$

where the bilinear forms $a(\cdot, \cdot), b(\cdot, \cdot)$ are given in (2.25) and

3.3. Lemma. *The bilinear form $b(\cdot, \cdot)$ satisfies the following inf-sup property: for any $\Phi \in \mathbb{V}_0$,*

$$(3.11) \quad \|D\Phi\|_{L^p(\Omega)} \leq C \sup_{0 \neq v_h \in \mathbb{V}_0} \frac{b(\Phi, v_h)}{\|Dv_h\|_{L^q(\Omega)}}.$$

Proof The proof of this fact involves noticing that the Ritz projection is in fact a Fortin operator, that is R satisfies

$$(3.12) \quad b(\Phi, v - Rv) = 0 \quad \forall v \in W^{1,q}(\Omega)$$

and that there exists a constant such that

$$(3.13) \quad \|D(Rv)\|_{L^q(\Omega)} \leq C \|Dv\|_{L^q(\Omega)}.$$

The orthogonality is clear in view of the definition of the projection. For the stability bound, we refer to [CT87]. Hence, using the inf-sup condition from Lemma 2.10 we have

$$(3.14) \quad \begin{aligned} \|D\Phi\|_{L^p(\Omega)} &\leq \sup_{0 \neq v \in W_0^{1,q}(\Omega)} \frac{b(\Phi, v)}{\|Dv\|_{L^q(\Omega)}} \\ &\leq \sup_{0 \neq v \in W_0^{1,q}(\Omega)} \frac{b(\Phi, Rv)}{\|Dv\|_{L^q(\Omega)}} \\ &\leq \sup_{0 \neq v_h \in \mathbb{V}_0} \frac{b(\Phi, v_h)}{\|Dv_h\|_{L^q(\Omega)}}, \end{aligned}$$

as required. \square

3.4. Theorem (existence and uniqueness of solution to (3.10)). *There exists a unique tuple $(u_h, w_h) \in \mathbb{V}_g \times \mathbb{V}$ solving (3.10). They satisfy the stability bound*

$$(3.15) \quad \|Du_h\|_{L^p(\Omega)} + \|w_h\|_{L^q(\Omega)}^{q-1} \leq C \left(\|\Delta g\|_{L^p(\Omega)} + \|Dg\|_{L^p(\Omega)} \right).$$

Note that since $g \in W^{2,\infty}(\Omega)$, the right hand side of (3.15) is finite.

Proof The proof of this mirrors that of Theorem 2.11. We begin by noting that for $\psi = w_h$ we have

$$(3.16) \quad a(w_h, w_h) + b(u_h, w_h) = f(w_h).$$

Now for $\phi = u_{h,0} := u_h - Rg$ we see that

$$(3.17) \quad b(w_h, u_h - Rg) = 0,$$

hence

$$(3.18) \quad \begin{aligned} f(w_h) &= a(w_h, w_h) + b(Rg, w_h) \\ &= a(w_h, w_h) + b(g, w_h) \end{aligned}$$

Now, by integrating by parts elementwise and writing everything explicitly, we obtain

$$(3.19) \quad \begin{aligned} \|w_h\|_{L^q(\Omega)}^q &= \int_{\Omega} \Delta g w_h \, d\mathbf{x} \\ &\leq \|\Delta g\|_{L^p(\Omega)} \|w_h\|_{L^q(\Omega)}, \end{aligned}$$

and hence

$$(3.20) \quad \|w_h\|_{L^q(\Omega)}^{q-1} \leq \|\Delta g\|_{L^p(\Omega)}.$$

The result follows because

$$(3.21) \quad \begin{aligned} \|Du_{h,0}\|_{L^p(\Omega)} &\leq C \sup_{0 \neq v_h \in \mathbb{V}_0} \frac{b(u_{h,0}, v_h)}{\|Dv_h\|_{L^q(\Omega)}} \\ &\leq C \left(\sup_{0 \neq v_h \in \mathbb{V}_0} \frac{b(u_h, v_h)}{\|Dv_h\|_{L^q(\Omega)}} + \sup_{0 \neq v_h \in \mathbb{V}_0} \frac{b(Rg, v_h)}{\|Dv_h\|_{L^q(\Omega)}} \right) \\ &\leq C \left(\sup_{0 \neq v_h \in \mathbb{V}_0} \frac{-a(w_h, v_h)}{\|Dv_h\|_{L^q(\Omega)}} + \sup_{0 \neq v_h \in \mathbb{V}_0} \frac{b(g, v_h)}{\|Dv_h\|_{L^q(\Omega)}} \right) \\ &\leq C \left(\| |w_h|^{q-1} \|_{L^p(\Omega)} + \|Dg\|_{L^p(\Omega)} \right) \\ &\leq C \left(\|w_h\|_{L^q(\Omega)}^{q-1} + \|Dg\|_{L^p(\Omega)} \right) \end{aligned}$$

by the discrete inf-sup condition in Lemma 3.3 and the same argument as in the proof of Theorem 2.11. Since

$$(3.22) \quad \|Du_h\|_{L^p(\Omega)} \leq \|Du_{h,0}\|_{L^p(\Omega)} + \|Dg\|_{L^p(\Omega)}$$

combining (3.19), (3.21) and (3.22) concludes the proof. \square

Next we state some technical properties that will be used in the theorem that follows.

3.5. Lemma (Properties of $a(\cdot, \cdot)$, cf. [San93, Prop 3.1]). *For any $p \geq 2$, there exist constants*

(1) $C_1 > 0$ such that

$$(3.23) \quad C_1 \frac{\|w - w_h\|_{L^q(\Omega)}^2}{\|w\|_{L^q(\Omega)}^{2-q} - \|w_h\|_{L^q(\Omega)}^{2-q}} \leq a(w, w - w_h) - a(w_h, w - w_h).$$

(2) $C_2 > 0$ such that

$$(3.24) \quad C_2 \int_{\Omega} \left| |w|^{q-2} w - |w_h|^{q-2} w_h \right| |w - w_h| \, d\mathbf{x} \leq a(w, w - w_h) - a(w_h, w - w_h).$$

(3) $C_3 > 0$ such that

$$(3.25) \quad a(w, w - Rw) - a(w_h, w - Rw) \leq C_3 \left(\int_{\Omega} \left| |w|^{q-2} w - |w_h|^{q-2} w_h \right| |w - w_h| \, d\mathbf{x} \right)^{1/p} \|w - Rw\|_{L^q(\Omega)}.$$

3.6. Theorem (Approximability of the numerical schemes). *Let $(u, w) \in \mathbf{W}_g^{k+1,p}(\Omega) \times \mathbf{W}^{k+1,q}(\Omega)$ be the unique solution of (2.24) and $(u_h, w_h) \in \mathbb{V}_g \times \mathbb{V}$ be the finite element approximation to (3.10). Then, the following error estimate holds true*

$$(3.26) \quad \begin{aligned} \|w - w_h\|_{L^q(\Omega)} &\leq Ch^{\frac{q}{2}(k+1)} |w|_{\mathbf{W}^{k+1,q}(\Omega)}^{q/2}, \\ \|Du - Du_h\|_{L^p(\Omega)} &\leq Ch^{\min(k, \frac{k+1}{p-1})} \left(|u|_{\mathbf{W}^{k+1,p}(\Omega)}^{q/p} + |w|_{\mathbf{W}^{k+1,q}(\Omega)}^{q/p} \right). \end{aligned}$$

Proof We begin by noting the Galerkin orthogonality results

$$(3.27) \quad \begin{aligned} b(w - w_h, \phi) &= 0 \quad \forall \phi \in \mathbb{V}, \\ a(w, \psi) - a(w_h, \psi) + b(u - u_h, \psi) &= 0 \quad \forall \psi \in \mathbb{V}, \end{aligned}$$

in view of (2.24) and (3.10). Taking $\psi = Rw - w_h$, where R denotes the Ritz projection, we see

$$(3.28) \quad \begin{aligned} 0 &= a(w, Rw - w_h) - a(w_h, Rw - w_h) + b(u - u_h, Rw - w_h) \\ &= a(w, Rw - w_h) - a(w_h, Rw - w_h) + b(u - u_h, Rw - w) + b(u - u_h, w - w_h), \end{aligned}$$

in view of the linearity of $b(\cdot, \cdot)$. Now by (3.27) and the definition of R we have

$$(3.29) \quad \begin{aligned} b(u - u_h, Rw - w) + b(u - u_h, w - w_h) &= b(u - u_h - R(u - u_h), Rw - w) + b(u - u_h - R(u - u_h), w - w_h) \\ &= b(u - u_h - R(u - u_h), Rw - w_h) \\ &= 0. \end{aligned}$$

Hence

$$(3.30) \quad 0 = a(w, Rw - w_h) - a(w_h, Rw - w_h),$$

and we see through the semilinearity of $a(\cdot, \cdot)$

$$(3.31) \quad \begin{aligned} a(w, w - w_h) - a(w_h, w - w_h) &= a(w, w - Rw) - a(w_h, w - Rw) + a(w, Rw - w_h) - a(w_h, Rw - w_h) \\ &= a(w, w - Rw) - a(w_h, w - Rw). \end{aligned}$$

Making use of this and Lemma 3.5

$$(3.32) \quad a(w, w - w_h) - a(w_h, w - w_h) \leq C_3 \left(\int_{\Omega} |w|^{q-2} w - |w_h|^{q-2} w_h |w - w_h| \, d\mathbf{x} \right)^{1/p} \|w - Rw\|_{L^q(\Omega)},$$

Now Young's inequality with ϵ states for $a, b, \epsilon > 0$

$$(3.33) \quad ab \leq \frac{1}{p}(\epsilon a)^p + \frac{1}{q} \left(\frac{b}{\epsilon} \right)^q$$

which, upon applying to (3.32), shows

$$(3.34) \quad a(w, w - w_h) - a(w_h, w - w_h) \leq \frac{C_3 \epsilon^p}{p} \int_{\Omega} |w|^{q-2} w - |w_h|^{q-2} w_h |w - w_h| \, d\mathbf{x} + \frac{C_3}{q \epsilon^q} \|w - Rw\|_{L^q(\Omega)}^q.$$

Now using the Lemma 3.5 again it is clear that

$$(3.35) \quad \frac{C_1 \|w - w_h\|_{L^q(\Omega)}^2}{2(\|w\|_{L^q(\Omega)}^{2-q} - \|w_h\|_{L^q(\Omega)}^{2-q})} + \frac{C_2}{2} \int_{\Omega} |w|^{q-2} w - |w_h|^{q-2} w_h |w - w_h| \, d\mathbf{x} \leq a(w, w - w_h) - a(w_h, w - w_h).$$

Now choosing $\epsilon = \left(\frac{C_2 p}{2C_3} \right)^{1/p}$ and combining (3.34) and (3.35) we have

$$(3.36) \quad \|w - w_h\|_{L^q(\Omega)}^2 \leq \frac{2C_3}{qC_1} \left(\frac{C_2 p}{2C_3} \right)^{\frac{1}{q-1}} (\|w\|_{L^q(\Omega)}^{2-q} - \|w_h\|_{L^q(\Omega)}^{2-q}) \|w - Rw\|_{L^q(\Omega)}^q,$$

and hence using the approximation properties of the Ritz projection from Definition 3.1

$$(3.37) \quad \|w - w_h\|_{L^q(\Omega)} \leq C \|w - Rw\|_{L^q(\Omega)}^{q/2} \leq Ch^{\frac{q}{2}(k+1)} |w|_{W^{k+1,q}(\Omega)}^{q/2}.$$

To show a bound for the primal variable we make use of the inf-sup condition from Lemma 3.3, noting that in view of Galerkin orthogonality and the definition of R that

$$(3.38) \quad \begin{aligned} 0 &= a(w, \psi) - a(w_h, \psi) + b(u - u_h, \psi) \\ &= a(w, \psi) - a(w_h, \psi) + b(Ru - u_h, \psi) \quad \forall \psi \in \mathbb{V}. \end{aligned}$$

It is then clear that

$$\begin{aligned}
\|D(Ru) - Du_h\|_{L^p(\Omega)} &\leq \sup_{0 \neq v_h \in \mathbb{V}_0} \frac{b(Ru - u_h, v_h)}{\|Dv_h\|_{L^q(\Omega)}} \\
&= \sup_{0 \neq v_h \in \mathbb{V}_0} \frac{a(w_h, v_h) - a(w, v_h)}{\|Dv_h\|_{L^q(\Omega)}} \\
(3.39) \quad &\leq C_3 \sup_{0 \neq v_h \in \mathbb{V}_0} \frac{\left(\int_{\Omega} |w|^{p-2} w - |w_h|^{p-2} w_h \right) |w - w_h| \, dx \Big)^{1/p}}{\|Dv_h\|_{L^q(\Omega)}} \\
&\leq C_3 C_P \left(\int_{\Omega} |w|^{p-2} w - |w_h|^{p-2} w_h \right) |w - w_h| \, dx \Big)^{1/p},
\end{aligned}$$

through the Poincaré inequality in Proposition 2.6. Now combining Lemma 3.5 and (3.31) we have

$$(3.40) \quad \int_{\Omega} |w|^{p-2} w - |w_h|^{p-2} w_h \Big) |w - w_h| \, dx \leq \left(\frac{C_3}{C_2} \right)^q \|w - Rw\|_{L^q(\Omega)}^q.$$

Substituting (3.40) into (3.39) results in

$$(3.41) \quad \|D(Ru) - Du_h\|_{L^p(\Omega)} \leq \left(\frac{C_3}{C_2} \right)^{q/p} C_3 C_P \|w - Rw\|_{L^q(\Omega)}^{q/p}.$$

The result follows from the fact

$$(3.42) \quad \|Du - Du_h\|_{L^p(\Omega)} \leq \|D(Ru) - Du_h\|_{L^p(\Omega)} + \|D(Ru) - Du\|_{L^p(\Omega)}$$

and using the approximation properties of the Ritz projection, concluding the proof. \square

3.7. Remark (Optimality of the bounds). Notice that the rates trail off as p gets large. A similar phenomena was noticed when constructing methods for the p -Laplacian [Cia78, Thm 5.3.5] where for a conforming piecewise linear approximation, u_h , the error behaved like

$$(3.43) \quad \|u - u_h\|_{W^{1,p}(\Omega)} \leq Ch^{1/(p-1)}.$$

An analysis based on quasi-norms [BL94] was then introduced to rectify this. It may be possible to use these techniques to show optimal error bounds for the p -Bilaplacian based on the quasi-norm

$$(3.44) \quad \|u\|_{v,p}^p := \int_{\Omega} |\Delta u|^2 (|\Delta u| + |\Delta v|)^{p-2}.$$

We shall not push this point further in this work however. Instead, in order to try to characterise the limiting problem, we shall focus on convergence under minimal regularity.

We begin by defining the semilinear form

$$(3.45) \quad c((u, w), (\phi, \psi)) := a(w, \psi) + b(u, \psi) + b(w, \phi),$$

then the discrete mixed form of the Bilaplacian can be written, equivalently to (3.10), as seeking $(u_h, w_h) \in \mathbb{V}_g \times \mathbb{V}$ such that

$$(3.46) \quad c((u_h, w_h), (\phi, \psi)) = f(\psi) \quad \forall (\phi, \psi) \in \mathbb{V}_0 \times \mathbb{V}.$$

3.8. Theorem (Convergence under minimal regularity). *Let (u_h, w_h) be a sequence of finite element solutions of (3.10) indexed by the mesh parameter h and let also $u \in W_g^{2,p}(\Omega)$ be the solution of the p -Bilaplacian. Then we have*

- $u_h \rightarrow u$ strongly in L^p as $h \rightarrow 0$,
- $w_h \rightharpoonup w$ weakly in L^q as $h \rightarrow 0$.

Proof The stability result given in Theorem 3.4 allows us to infer that the sequence (u_h, w_h) is bounded uniformly in h . This means, up to a subsequence, that there exists a $(u^*, w^*) \in W_g^{1,p}(\Omega) \times L^q(\Omega)$ such that $u_h \rightarrow u^*$ strongly in $L^p(\Omega)$ and $w_h \rightharpoonup w^*$ weakly in $L^q(\Omega)$.

Now suppose $v_1 \in C^\infty(\Omega)$. Take $(\phi, \psi) = (0, Rv_1)$ in (3.46). Then,

$$(3.47) \quad 0 = c((u_h, w_h), (0, Rv_1)) = a(w_h, Rv_1) + b(u_h, Rv_1).$$

Since $u_h \rightarrow u^*$ and by the properties of the projection R given in Definition 3.1 we have that

$$(3.48) \quad b(u_h, Rv_1) \rightarrow b(u^*, v_1).$$

Also, since $w_h \rightarrow w^*$ and $Rv_1 \rightarrow v_1$ strongly we have

$$(3.49) \quad a(w_h, Rv_1) \rightarrow a(w^*, v_1).$$

Hence

$$(3.50) \quad a(w_h, Rv_1) + b(u_h, Rv_1) \rightarrow a(w^*, v_1) + b(u^*, v_1).$$

Now suppose $v_2 \in C^\infty(\Omega)$ and take $(\phi, \psi) = (Rv_2, 0)$ in (3.46), then

$$(3.51) \quad f(Rv_2) = b(w_h, Rv_2).$$

By the same arguments we have

$$(3.52) \quad b(w_h, Pv_2) - f(Pv_2) \rightarrow b(w^*, v_2) - f(v_2).$$

Using density of $C^\infty(\Omega) \times C^\infty(\Omega)$ functions in $W^{1,p}(\Omega) \times L^q(\Omega)$ shows that (u^*, w^*) must solve the Bilaplacian and since the solution was unique, the whole sequence $(u_h, w_h) \rightarrow (u, w)$. \square

3.9. Theorem. *Let $u_{h,p} \in \mathbb{V}_g$ be the Galerkin solution of (3.10) and let u_∞ denote a candidate ∞ -Biharmonic function then along a subsequence we have*

$$(3.53) \quad u_{h,p_j} \rightarrow u_\infty \in C^0 \text{ as } p \rightarrow \infty \text{ and } h \rightarrow 0.$$

Proof The proof consists of combining Theorems 2.12 and 3.8 and noticing that

$$(3.54) \quad \|u_{h,p_j} - u_\infty\|_{C^0(\Omega)} \leq \|u_{h,p_j} - u_{p_j}\|_{C^0(\Omega)} + \|u_{p_j} - u_\infty\|_{C^0(\Omega)},$$

as required. \square

3.10. Corollary. *If $d = 1$, then there exists a unique \mathcal{D} -solution u_∞ to the ∞ -Bilaplacian on Ω and along a subsequence of indices we have*

$$(3.55) \quad u_{u,p_j} \rightarrow u_\infty \in C^0(\bar{\Omega}) \text{ as } p \rightarrow \infty \text{ and } h \rightarrow 0.$$

4. NUMERICAL EXPERIMENTS

In this section we summarise numerical experiments validating the analysis done in previous sections.

4.1. Benchmarking. We begin by benchmarking the scheme against a known solution of the p -Biharmonic problem. To do this we introduce a source term into the problem

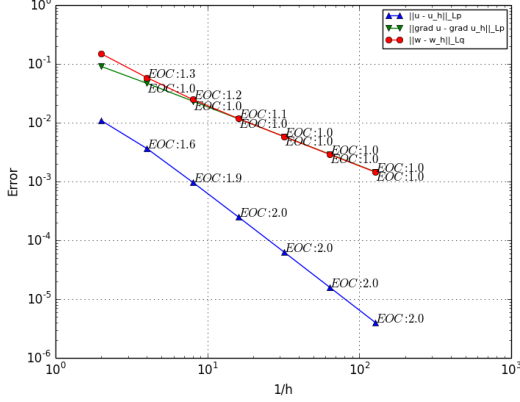
$$(4.1) \quad \begin{cases} \Delta(|\Delta u|^{p-2} \Delta u) = f, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \\ Du = Dg, & \text{on } \partial\Omega. \end{cases}$$

This allows us to pick a function g and construct the appropriate source term such that g solves (4.1). For these tests we choose

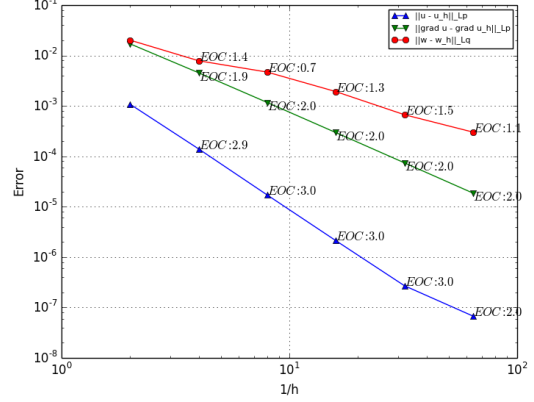
$$(4.2) \quad u(x, y) = \sin(\pi x) \sin(\pi y).$$

We take $\Omega = [-1, 1]^2$ and discretise the domain with a sequence of concurrently refined criss-cross type meshes.

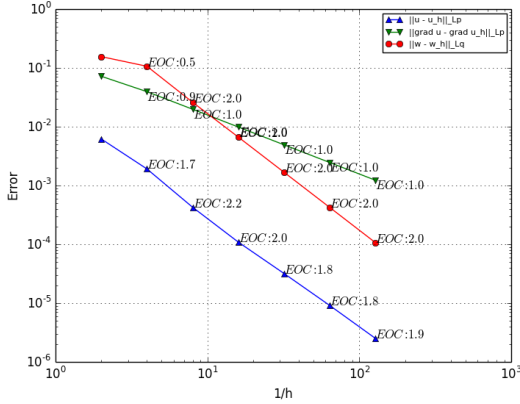
FIGURE 1. Benchmarking results for the mixed finite element approximation to (4.1). We test the cases $p = 2$ and $p = 4$ as well for polynomials of degree $k = 1$ and $k = 2$. The results show that in the case $p = 2$ the convergence rates as predicted in the analysis are achieved. In the case $p > 2$ convergence rates are both higher than predicted for both primal *and* auxiliary variable. Notice also that when $p > 2$ the auxiliary variable converges faster than the case $p = 2$.



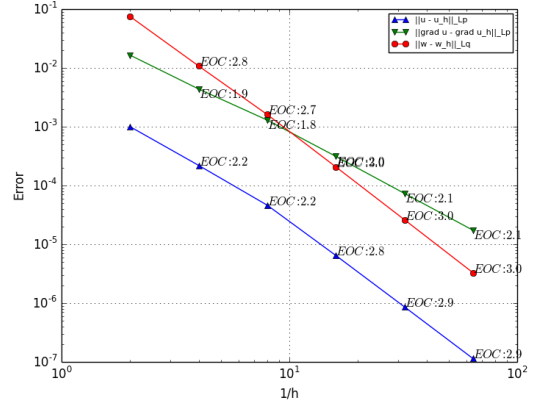
(a) The piecewise linear finite element approximation to the 2-Bilaplacian.



(b) The piecewise quadratic finite element approximation to the 2-Bilaplacian.



(c) The piecewise linear finite element approximation to the 4-Bilaplacian.



(d) The piecewise quadratic finite element approximation to the 4-Bilaplacian.

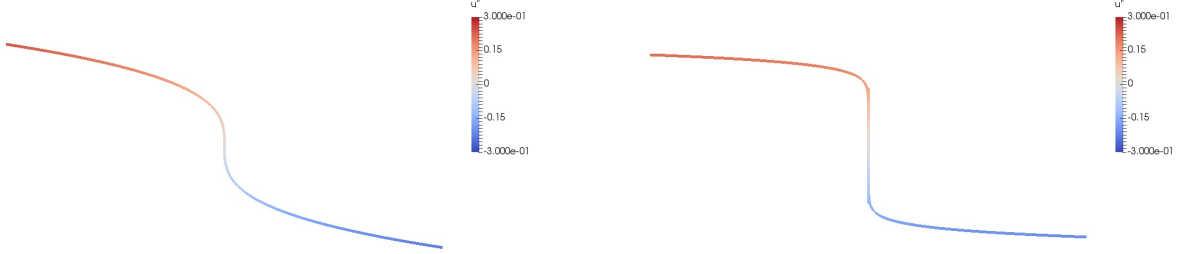
4.2. Characterising ∞ -Harmonic functions. In this section we illustrate some of the properties of ∞ -Biharmonic functions. The results illustrate that for practical purposes, as one would expect, the approximation of p -Biharmonic functions for large p gives good resolution of candidate ∞ -Biharmonic functions.

Test 1: the 1-dimensional problem. We consider the Dirichlet problem for the p -Bilaplacian for $d = 1$ with the boundary data given by the values of the cubic function

$$(4.3) \quad g(x) = \frac{1}{120}(4x - 3)(2x - 1)(4x - 1)$$

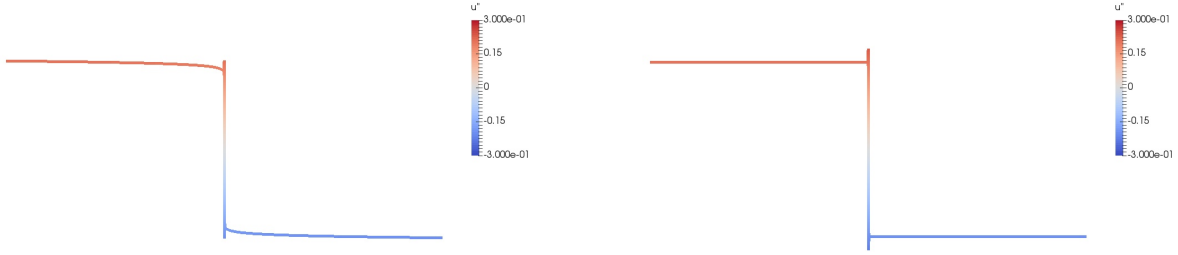
on $(0, 1)$. We simulate the p -Bilaplacian for increasing values of p and present the results in Figure 2 indicating that in the limit the ∞ -Biharmonic function should be piecewise quadratic.

FIGURE 2. A mixed finite element approximations to an ∞ -Biharmonic function using p -Biharmonic functions for various p for the problem given by (4.3). Notice that as p increases, u'' tends to a piecewise constant up to Gibbs oscillations. This is an indication the solution is indeed piecewise quadratic. Also there is only one breaking point in the solution, the location and size of this discontinuity was fully characterised in [KP16b].



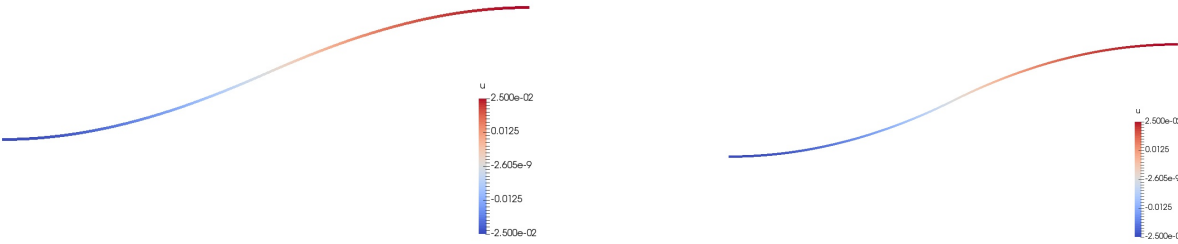
(a) The approximation to u'' , the Laplacian of the solution of the 4-Bilaplacian.

(b) The approximation to u'' , the Laplacian of the solution of the 12-Bilaplacian.



(c) The approximation to u'' , the Laplacian of the solution of the 42-Bilaplacian.

(d) The approximation to u'' , the Laplacian of the solution of the 202-Bilaplacian.



(e) The approximation to u , the solution of the 4-Bilaplacian.

(f) The approximation to u , the solution of the 202-Bilaplacian.

Test 2: the 2-dimensional problem. Now we illustrate some of the complicated behaviour of the p -Bilaplacian for $n = 2$:

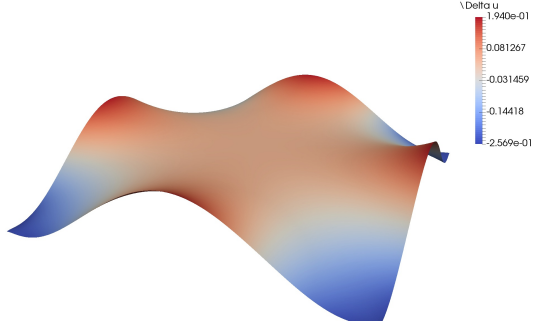
$$(4.4) \quad \left\{ \begin{array}{ll} \Delta(|\Delta u|^{p-2}\Delta u) = 0, & \text{in } \Omega = [-1, 1]^2, \\ u = g, & \text{on } \partial\Omega, \\ Du = Dg, & \text{on } \partial\Omega, \end{array} \right.$$

where g is prescribed as

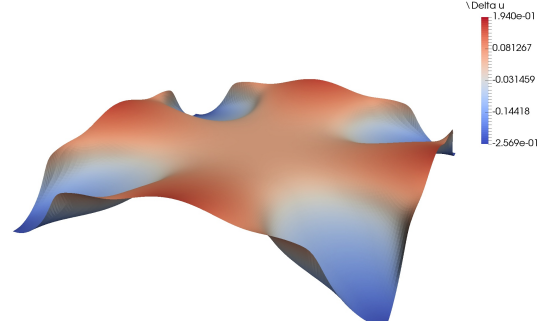
$$(4.5) \quad g(x, y) = \frac{1}{m20} \cos(m\pi x) \cos(m\pi y),$$

for various values of m . We simulate the p -Bilaplacian for increasing values of p and present the results in Figures 3,4 and 5 indicating that in the limit the ∞ -Biharmonic function should be piecewise quadratic however the behaviour is quite unexpected and complicated interface patterns emerge even with this relatively simple boundary data.

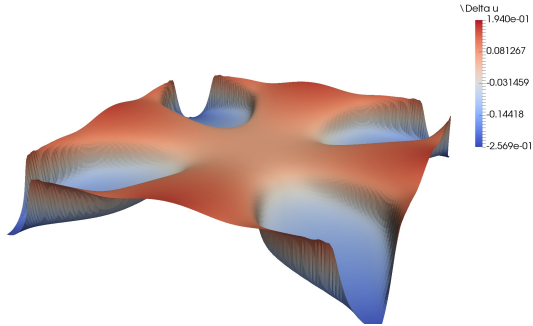
FIGURE 3. A mixed finite element approximations to an ∞ -Biharmonic function using p -Biharmonic functions for various p for the problem given by (4.4) and (4.5) with $m = 1$. Notice that as p increases, Δu tends to be piecewise constant. This is an indication the solution satisfies the Poisson equation with piecewise constant right hand side albeit with an extremely complicated solution pattern that clearly warrants further investigation.



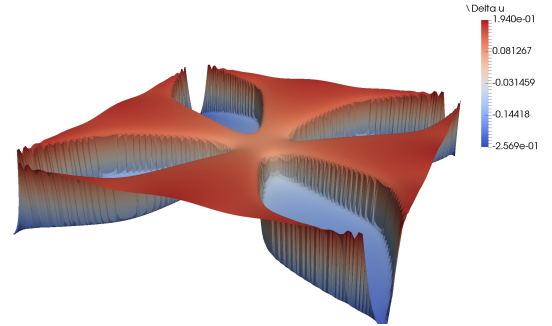
(a) The approximation to Δu , the Laplacian of the solution of the 4-Bilaplacian.



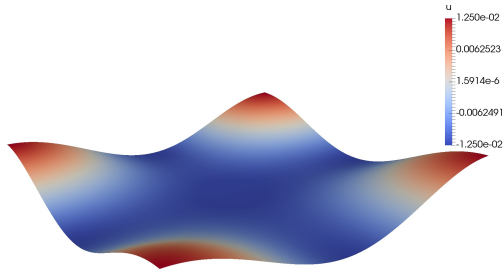
(b) The approximation to Δu , the Laplacian of the solution of the 42-Bilaplacian.



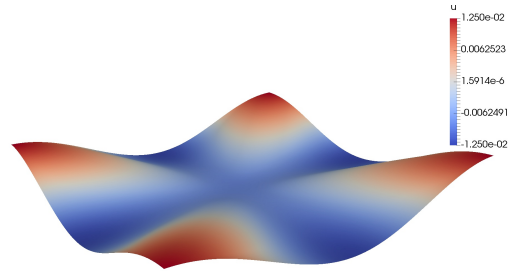
(c) The approximation to Δu , the Laplacian of the solution of the 68-Bilaplacian.



(d) The approximation to Δu , the Laplacian of the solution of the 142-Bilaplacian.

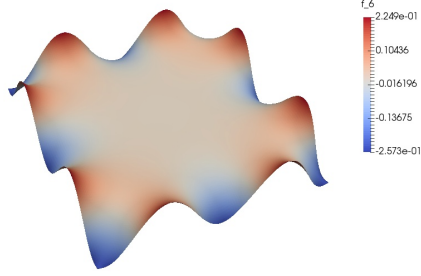


(e) The approximation to u , the solution of the 4-Bilaplacian.

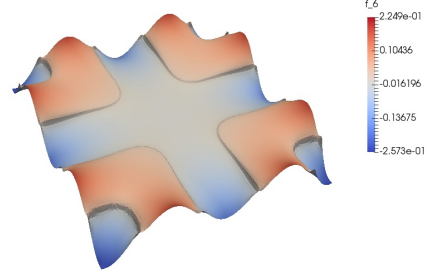


(f) The approximation to u , the solution of the 142-Bilaplacian.

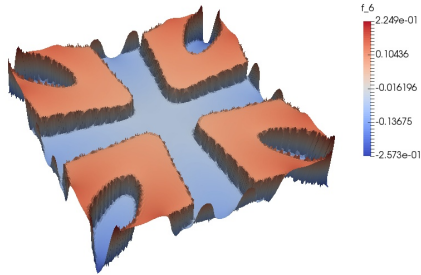
FIGURE 4. A mixed finite element approximations to an ∞ -Biharmonic function using p -Biharmonic functions for various p for the problem given by (4.4) and (4.5) with $m = 2$. Notice that as p increases, Δu tends to be piecewise constant. This is an indication the solution satisfies the Poisson equation with piecewise constant right hand side albeit with an extremely complicated solution pattern that clearly warrants further investigation.



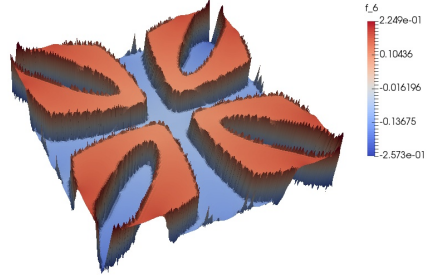
(a) The approximation to Δu , the Laplacian of the solution of the 4-Bilaplacian.



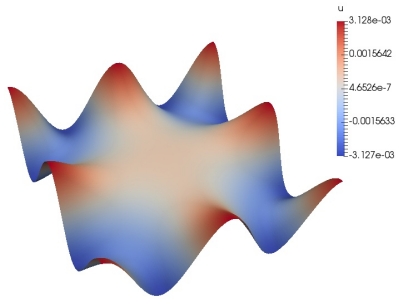
(b) The approximation to Δu , the Laplacian of the solution of the 42-Bilaplacian.



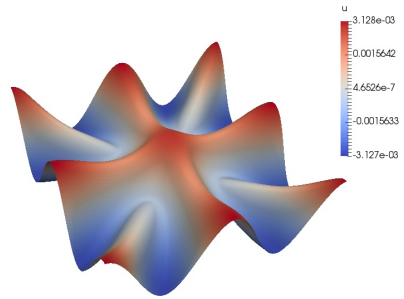
(c) The approximation to Δu , the Laplacian of the solution of the 68-Bilaplacian.



(d) The approximation to Δu , the Laplacian of the solution of the 142-Bilaplacian.



(e) The approximation to u , the solution of the 4-Bilaplacian.

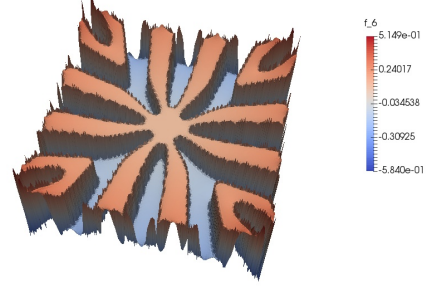


(f) The approximation to u , the solution of the 142-Bilaplacian.

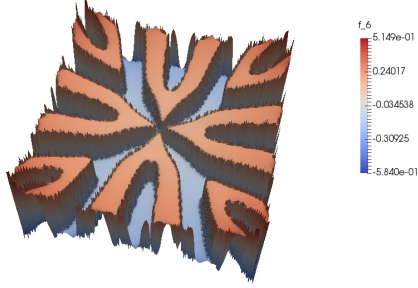
FIGURE 5. A mixed finite element approximations to an ∞ -Biharmonic function using p -Biharmonic functions for various p for the problem given by (4.4) and (4.5) with $m = 3$. Notice that as p increases, Δu tends to be piecewise constant. This is an indication the solution satisfies the Poisson equation with piecewise constant right hand side albeit with an extremely complicated solution pattern that clearly warrants further investigation.



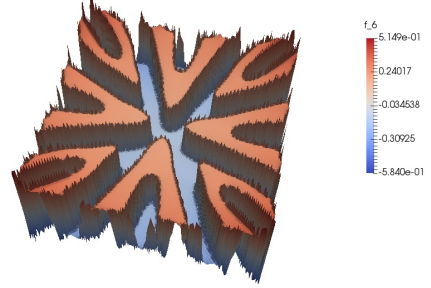
(a) The approximation to Δu , the Laplacian of the solution of the 4-Bilaplacian.



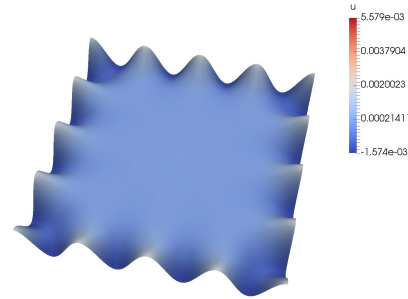
(b) The approximation to Δu , the Laplacian of the solution of the 42-Bilaplacian.



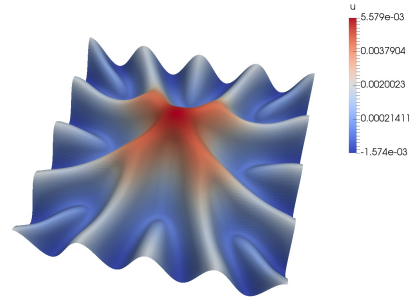
(c) The approximation to Δu , the Laplacian of the solution of the 68-Bilaplacian.



(d) The approximation to Δu , the Laplacian of the solution of the 142-Bilaplacian.



(e) The approximation to u , the solution of the 4-Bilaplacian.



(f) The approximation to u , the solution of the 142-Bilaplacian.

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NIKOS KATZOURAKIS DEPARTMENT OF MATHEMATICS AND STATISTICS, WHITEKNIGHTS, UNIVERSITY OF READING, READING RG6 6AX, UK N.Katzourakis@reading.ac.uk.

TRISTAN PRYER DEPARTMENT OF MATHEMATICS AND STATISTICS, WHITEKNIGHTS, UNIVERSITY OF READING, READING RG6 6AX, UK T.Pryer@reading.ac.uk.